The Yoneda Embedding and Group Representations

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I will be describing the relationship between group representations (for which [1] is an excellent introduction) and functor categories. Furthermore, I will use this relationship to give a way to construct some group representations purely categorically. The main tool in achieving this is the Yoneda embedding, an embedding derived from the Yoneda Lemma [3].

Definition 0.1. The **Yoneda embedding** is a fully faithful embedding

$$y\colon C \hookrightarrow [C^{\mathrm{op}}, \mathbf{Set}]$$

such that for objects c, c',

$$y(c)(c') \coloneqq C(c', c)$$

is natural in c and c'. For a morphism $g\colon x \to y$ in C we have,

$$y(g)(c') \coloneqq C(c', g)\colon C(c', x) \to C(c', y)$$

which sends $f\colon c' \to x$ to $g \circ f$. For a morphism $h\colon y \to x$ in C^{op} we similarly have,

$$y(c)(h) \coloneqq C(h, c)\colon C(x, c) \to C(y, c)$$

which sends $f\colon x \to c$ to $f \circ h$. This generalises to an enriched Yoneda embedding [2] where for a V-enriched category C, if V has underlying set structure, we have the embedding,

$$y\colon C \hookrightarrow [C^{\mathrm{op}}, V]; \quad y(c)(c') \coloneqq C(c', c);$$
$$y(g)(c') \coloneqq C(c', g); \quad y(c)(h) \coloneqq C(h, c).$$

Definition 0.2. Let G be a group. The $\mathbf{Vect}_{\mathbb{C}}$-enriched category $B\mathbb{C}G$ has one object $\bullet$ and hom object $B\mathbb{C}G(\bullet, \bullet) = \mathbb{C}G$. The composition $\circ\colon \mathbb{C}G \otimes \mathbb{C}G \to \mathbb{C}G$ is given by the group multiplication $g \otimes h \mapsto g \cdot h$. The identity morphism $\mathrm{id}\colon \mathbb{C} \to B\mathbb{C}G$ sends $c \in \mathbb{C}$ to its product ce with the group identity.

Definition 0.3. Let G be a group and V be a vector space. A **G-representation** on V is a group homomorphism $\rho\colon G \to GL(V)$.

Example 0.4. A particular pair of examples of a representation we will be interested in is the **right and left regular representations**. For any group G the group algebra $\mathbb{C}G$ over $\mathbb{C}$ can be considered as a vector space. Multiplication on the left by an element $g \in G$ is, by definition of the algebra multiplication, a linear map

$$(g\cdot)\colon \mathbb{C}G \to \mathbb{C}G;\ \sum_{h\in G} c_h h \mapsto \sum_{h\in G} c_h (g\cdot h).$$

We can quickly check this gives a group homomorphism $\rho\colon G \to GL(\mathbb{C}G)$ by $g \mapsto (g\cdot)$. Multiplication can either be done on the left, as above, or on the right and this gives rise to the left regular representation

$${}^{\mathrm{reg}}\mathbb{C}G := (\mathbb{C}G, \rho\colon G \to \mathbb{C}G; g \mapsto g\cdot(-))$$

or the right regular representation

$$\mathbb{C}G^{\mathrm{reg}} := (\mathbb{C}G, \rho\colon G \to \mathbb{C}G; g \mapsto (-)\cdot g)$$

respectively.

Definition 0.5. Let $\rho\colon G \to GL(V)$ and $\tau\colon G \to GL(W)$ be group representations. A G-**equivariant map** is a linear map $f\colon V \to W$ such that the following diagram commutes for all $g \in G$.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle\rho(g)}\downarrow & & \downarrow{\scriptstyle\tau(g)} \\ V & \xrightarrow{f} & W \end{array}$$

Definition 0.6. The category $\mathrm{Rep}(G)$ has objects which are pairs $(V, \rho\colon G \to GL(V))$ of a vector space V and a G-representation on V. The morphisms are given by the G-equivariant maps.

Definition 0.7. Let $I\colon [B\mathbb{C}G, \mathbf{Vect}_{\mathbb{C}}] \to \mathrm{Rep}(G)$ be the functor which sends a $\mathbf{Vect}_{\mathbb{C}}$-functor $F\colon B\mathbb{C}G \to \mathbf{Vect}_{\mathbb{C}}$ to the pair

$$(F(\bullet), \rho\colon G \to GL(F(\bullet))\,;\ g \mapsto F(g)) \in \mathrm{Rep}(G)$$

and sends a natural transformation η to the G-equivariant map which is the linear map corresponding to the component of η at $\bullet$.

To know this is indeed a functor we need to verify that $I(F)(g)$ is in $GL(F(\bullet))$ but since functors preserve isomorphism and all $g \in G$ have inverse g^{-1} then all $g \in G$ are isomorphisms of $B\mathbb{C}G$ and so we know that $F(g)$ is in $GL(F(\bullet))$. We also need to verify that a natural transformation does indeed map to a G-equivariant map but simply applying our functor to all morphisms and objects of a natural transformation square gives the required condition on a G-equivariant map.

Lemma 0.8. *The functor* $I\colon [B\mathbb{C}G, \mathbf{Vect}_{\mathbb{C}}] \to \mathrm{Rep}(G)$ *is an isomorphism of categories.*

Proof. Let $J\colon \mathrm{Rep}(G) \to [B\mathbb{C}G, \mathbf{Vect}_{\mathbb{C}}]$ be the functor which sends $(V, \rho\colon G \to GL(V)) \in \mathrm{Rep}(G)$ to the functor

$$J(\rho)\colon B\mathbb{C}G \to \mathbf{Vect}_{\mathbb{C}};\ \bullet \mapsto V, \sum_{g\in G} c_g g \mapsto \sum_{g\in G} c_g \rho(g)$$

and sends a G-equivariant map $f\colon V \to V$ to the natural transformation with component f at $\bullet$ considered as a linear map.

We claim J is inverse to I. Firstly if $F\colon B\mathbb{C}G \to \mathbf{Vect}_{\mathbb{C}}$ is a functor then the representation of $I(F)$ is the restriction of F on G which sends $g \in G$ to $F(g) \in GL(F(\bullet))$ so $I(F)$ is the pair $(F(\bullet), F\colon G \to GL(F(\bullet))) \in \mathrm{Rep}(G)$. By definition of J we send $I(F)$ to the functor $J(I(F))$ which sends $\bullet$ to $F(\bullet)$ and $\sum_{g\in G} c_g g \in \mathbb{C}G$ to $\sum_{g\in G} c_g F(g)$. Because F is a $\mathbf{Vect}_{\mathbb{C}}$-functor, F is a linear map and so we have $\sum_{g\in G} c_g F(g) = F\left(\sum_{g\in G} c_g g\right)$ and hence $J \circ I(F) = F$.

Let $\eta\colon F \to G$ be a natural transformation in $[B\mathbb{C}G, \mathbf{Vect}_{\mathbb{C}}]$ with component $\eta_\bullet$ at $\bullet$. The G-equivariant map $I(\eta)$ is defined to be $\eta_\bullet$ and $J(I(\eta))$ is the natural transformation with component $\eta_\bullet$. Since η only has one component then this means $J \circ I(\eta) = \eta$. This shows $J \circ I = \mathrm{id}_{[B\mathbb{C}G, \mathbf{Vect}_{\mathbb{C}}]}$

Conversely let $\rho\colon G \to GL(V)$ be a G-representation. We know $J(\rho)$ is a functor such that $J(\rho)(\bullet) = V$ and for all $g \in G$ we have $J(\rho)(g) = \rho(g)$. Since $GL(J(\rho)(\bullet)) = GL(V)$ this means $I \circ J(\rho)$ is a G-representation $G \to GL(V)$ which sends g to $J(\rho)(g) = \rho(g)$ and hence $I \circ J(\rho) = \rho$.

Finally let f be a G-equivariant map then $J(f)$ is a natural transformation with single component f. Then I sends $J(f)$ back to f since f is the component of $J(f)$ at $\bullet$. Hence we have that $I \circ J = \mathrm{id}_{\mathrm{Rep}(G)}$ and so $J = I^{-1}$ as required. □

Theorem 0.9. *Consider the Yoneda embedding* $y\colon B\mathbb{C}G^{\mathrm{op}} \to [B\mathbb{C}G, \mathbf{Vect}_{\mathbb{C}}]$ *and the canonical isomorphism* $I\colon [B\mathbb{C}G, \mathbf{Vect}_{\mathbb{C}}] \to \mathrm{Rep}(G)$. *The following characterises* $I \circ y\colon B\mathbb{C}G^{\mathrm{op}} \to \mathrm{Rep}(G)$.

1. *The object* $\bullet$ *of* $B\mathbb{C}G^{\mathrm{op}}$ *is sent to the left regular representation* ${}^{\mathrm{reg}}\mathbb{C}G$.
2. *For any* $g \in \mathbb{C}G$ *we have* $(g\colon \bullet \to \bullet) \mapsto (g \cdot v\colon {}^{\mathrm{reg}}\mathbb{C}G \to {}^{\mathrm{reg}}\mathbb{C}G)$ *i.e. the* G*-equivariant map given by left multiplication by* g.

Proof.

1. Since $y(\bullet)$ is a functor then we need to find what $y(\bullet)$ does on objects and morphisms of $B\mathbb{C}G$ respectively. First $y(\bullet)(\bullet) = B\mathbb{C}G^{\mathrm{op}}(\bullet, \bullet)$ by Definition 0.1 of the Yoneda embedding. By Definition 0.2 we have $B\mathbb{C}G^{\mathrm{op}}(\bullet, \bullet) = \mathbb{C}G$ as a vector space and so we have the map of $y(\bullet)$

on objects. For the map on morphisms of $B\mathbb{C}G$ Definition 0.1 of the Yoneda embedding tells us that

$$y(\bullet)(g) = B\mathbb{C}G^{\text{op}}(g, \bullet) \colon B\mathbb{C}G^{\text{op}}(\bullet, \bullet) \to B\mathbb{C}G^{\text{op}}(\bullet, \bullet)$$

sends $v \in B\mathbb{C}G$ to $v \circ g$. The composition in $B\mathbb{C}G$ sends $v \circ g$ to the flipped multiplication gv and so $y(\bullet)$ is a functor sending g to the map $g \cdot (-) \in \text{End}(\mathbb{C}G)$. By Definition 0.7, the functor I sends $y(\bullet)$ to the pair

$$\begin{aligned}&(y(\bullet)(\bullet), \rho \colon G \to GL(y(\bullet)(\bullet); g \mapsto y(\bullet)(g)) \\ &= (\mathbb{C}G, \rho \colon G \to \mathbb{C}G; g \mapsto g \cdot (-))\end{aligned}$$

in $\text{Rep}(G)$. This is exactly the left regular representation ${}^{\text{reg}}\mathbb{C}G$.

2. First we need to calculate the natural transformation $y(g)$. Since g is a morphism $\bullet \to \bullet$ we know $y(g)$ is a natural transformation $y(\bullet) \to y(\bullet)$. Since $y(g)$ is a map of functors of $B\mathbb{C}G^{\text{op}}$ we can evaluate everything at $\bullet$. That is $y(g)(\bullet) \colon y(\bullet)(\bullet) \to y(\bullet)(\bullet)$ which is given by Definition 0.1 of the Yoneda embedding as the map $B\mathbb{C}G^{\text{op}}(\bullet, g)$ which sends $v \in B\mathbb{C}G^{\text{op}}(\bullet, \bullet)$ to $g \circ v$. As before the composition in $B\mathbb{C}G$ sends $g \circ v$ to the flipped multiplication vg, i.e. $y(g)(\bullet)$ is the linear map $(-) \cdot g$. Since $B\mathbb{C}G^{\text{op}}$ has only one object so $(-) \cdot g$ is the one component (linear map) of the natural transformation $y(g)$. By Definition 0.7, the functor I sends $y(g)$ map to the linear map $(-) \cdot g$ of vector spaces considered as G-equivariant map. That is $y(g)$ is precisely the G-equivariant map $(-) \cdot g \colon {}^{\text{reg}}\mathbb{C}G \to {}^{\text{reg}}\mathbb{C}G$.

□

References

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[3] Emily Riehl. In: *Category Theory in Context*. Dover Publications, 2017, pp. 57–60. ISBN: 9780486820804. URL: `https://math.jhu.edu/~eriehl/context.pdf`.